4. LÖTSTEDT P., Coulomb friction in two-dimensional rigid body systems, Z. Angew. Math. und Mech., 61, 12, 1981.
5. BOLOTOV E.A., On the motion of a material plane figure, constrained by connections with friction, Universitetskaya tip., Moscow, 1906.
6. SUSLOV G.K., Theoretical mechanics (Teoreticheskaya mekhanika), Gostekhizdat, Moscow-Leningrad, 1946.
7. APPEL' P., Theoretical mechanics (Teoreticheskaya mekhanika), Fizmatgiz, Moscow, 1960.
8. BOLOTOV E.A., On the impact of two rigid bodies under the action of friction, Izv. Mosk. inzh. uchilishcha, 2, 2, 1908.
9. PONTTRYAGIN L.S., Ordinary differential equations (Obyknovennye differentsial'nye uravneniya), Nauka, Moscow, 1982.

Translated by D.E.B.

PMM U.S.S.R., Vol.50, No.5, pp.550-558, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00
©1987 Pergamon Journals Ltd.

INTEGRABLE CASES OF THE HAMILTON-JACOBI EQUATIONS AND DYNAMIC SYSTEMS REDUCIBLE TO CANONICAL FORM*

A.A. BEKOV

Novel integrable cases of the Hamilton-Jacobi (HJ) equations are obtained. A method of reducing a class of non-autonomous dynamic systems to canonical form is given, and cases of their integrability are indicated. Comparison theorems are presented enabling the integrability of a dynamic system to be determined by observing the form of its Hamiltonian. The case of two bodies of variable mass in a resisting and gravitating medium are studied as an example.

1. The integration of canonical equations of motion is reduced to finding the complete integral of the corresponding HJ equation. The most interesting cases from the point of view of practical applications are the cases of integrability of the HJ, Liouville and Stäckel equations /1/ and their generalizations /2/. We shall establish new cases of integrability of the HJ equation of the form

$$p+\frac{1}{2}\sum_{i=1}^{n} g^{ii}p_i^2+\sum_{i=1}^{n} h^i p_i-U=0 \quad \left(p=\frac{\partial V}{\partial t},\quad p_i=\frac{\partial V}{\partial q_i}\right) \tag{1.1}$$

which generalize the result obtained by Yarov-Yarovoi /2/ and include the cases of integrability of Demin /3/, Liouville and Stäckel /1/.

Theorem 1.1. If the Hamiltonian is given by the formula

$$H=\frac{1}{2}\frac{\gamma}{b}\sum_{i=1}^{n}\frac{1}{a_i(q_i)}\left(p_i-\sum_{j=1}^{k}\varphi_j\frac{\partial\Phi_j}{\partial q_i}\right)^2- \sum_{j=1}^{k}\sigma_j\Phi_j-\frac{\gamma}{b}\sum_{i=1}^{n}U_i(q_i)+\Phi_0(t) \tag{1.2}$$

$$b=\sum_{i=1}^{n}b_i(q_i),\quad \Phi_j=\frac{1}{b}\sum_{i=1}^{n}\Phi_{ij}(q_i)\quad (j=1,2,\ldots,k) \tag{1.3}$$

$$\sigma_j=\varphi_j^{\cdot}-c_j\gamma\quad (j=1,2,\ldots,k\leqslant n) \tag{1.4}$$

where $a_i, b_i, U_i, \Phi_0, \Phi_{ij}$ are arbitrary continuous functions and $a_i\neq 0$, $b\neq 0$ and Φ_j are differentiable functions of the variables q_i, γ, σ_j, φ_j are continuous functions of time and c_j are arbitrary constants, then the HJ equation has a complete integral

---

*Prikl. Matem. Mekhan., 50, 5, 717-726, 1986

$$V=\sum_{j=1}^{k}\varphi_j\Phi_j-\int(h\gamma+\Phi_0)\,dt+W \tag{1.5}$$

$$W=\sum_{i=1}^{n}\int\Big[2a_i\Big(U_i+hb_i-\sum_{j=1}^{k}c_j\Phi_{ij}+\alpha_i\Big)\Big]^{1/2}dq_i \tag{1.6}$$

where h and α_i are arbitrary constants and

$$\alpha_1+\alpha_2+\ldots+\alpha_n=0 \tag{1.7}$$

Proof. We will seek a solution of the equation

$$\frac{\gamma}{b}\sum_{i=1}^{n}\Big[\frac{1}{2a_i}\Big(\frac{\partial V}{\partial q_i}-\sum_{j=1}^{k}\varphi_j\frac{\partial\Phi_j}{\partial q_i}\Big)^2-U_i\Big]-\sum_{j=1}^{k}\sigma_j\Phi_j+\Phi_0+\frac{\partial V}{\partial t}=0$$

in the form (1.5) where W is an unknown functions of $q_1, q_2, \ldots, q_n$. By virtue of (1.3), (1.4) we arrive at the equation

$$\sum_{i=1}^{n}\Big[\frac{1}{2a_i}\Big(\frac{\partial W}{\partial q_i}\Big)^2-U_i+\sum_{j=1}^{k}c_j\Phi_{ij}-hb_i\Big]=0$$

which has the solution (1.6) where the constants α_i satisfy condition (1.7). It remains to confirm the condition

$$\operatorname{Det}\|\partial^2V/\partial q_i\partial\alpha_j\|\neq 0 \tag{1.8}$$

We have

$$\operatorname{Det}\Big\|\frac{\partial^2V}{\partial q_i\partial\alpha_j}\Big\|=b\prod_{i=1}^{n}a_i\Big(p_i-\sum_{j=1}^{k}\varphi_j\frac{\partial\Phi_j}{\partial q_i}\Big)^{-1}\neq 0$$

since the functions φ_j and $\partial\Phi_j/\partial q_i$ are continuous and according to the condition $a_i\neq 0,\ b\neq 0$, which completes the proof.

Corollaries. 1°. If $c_j=0\ (j=1,2,\ldots,l\leqslant k)$ in (1.4), i.e. $\sigma_j=\varphi_j^{\cdot}$ and $\Phi_j(q)$ are arbitrary continuous functions, then the summation over j in (1.6) is carried out from $j=l+1$ to $j=k$.

2°. If all $c_j=0$, then $\Phi_0=0,\ \Phi_1=\Phi,\ \Phi_j=0\ (j=2,\ldots,k)$ yields the integrable case of /4/.

3°. Putting in (1.2) all $\Phi_j=0$, we obtain the integrable case of /2/, and assuming also that $\gamma=\text{const}$ and $\Phi_0=0$, we arrive at Liouville's theorem /1/.

Theorem 1.2. Let $n(n+1)$ functions $\varphi_{ij}(q_i)$ and $U_i(q_i)\ (i,j=1,2,\ldots,n)$ be given, for which the determinant $\Delta=\|\varphi_{ij}(q_i)\|\not\equiv 0$, as well as the continuous differentiable functions $\Phi_j(q_1,q_2,\ldots,q_n)\ (j=1,2,\ldots,k)$ and continuous function of time $\gamma, \varphi_j, \sigma_j, \Phi_0\ (j=1,2,\ldots,k)$. Then, provided that the Hamiltonian is given by the formula

$$H=\frac{1}{2}\gamma\sum_{i=1}^{n}\Big[\frac{A_i}{a_i}\Big(p_i-\sum_{j=1}^{k}\varphi_j\frac{\partial\Phi_j}{\partial q_i}\Big)^2-A_ib_i\Big]- \tag{1.9}$$
$$\sum_{j=1}^{k}\sigma_j\Phi_j-\gamma\sum_{i=1}^{n}A_iU_i+\Phi_0$$

$$A_i=\frac{1}{\Delta}\frac{\partial\Delta}{\partial\varphi_{i1}},\quad \Phi_j=\sum_{i=1}^{n}A_i\Phi_{ij}(q_i)\quad\begin{pmatrix}i=1,2,\ldots,n\\ j=1,2,\ldots,k\end{pmatrix} \tag{1.10}$$

$$\sigma_j=\varphi_j^{\cdot}-c_j\gamma\quad(j=1,2,\ldots,k\leqslant n) \tag{1.11}$$

and every coefficient a_i, b_i, Φ_{ij} depends only on the corresponding q_i and $a_i\neq 0$, the HJ equation has a complete integral of the form (1.5), where

$$W=\sum_{i=1}^{n}\int\Big[a_i\Big(b_i+2U_i+2h\varphi_{i1}-2\sum_{j=1}^{k}c_j\Phi_{ij}+\sum_{j=2}^{n}\alpha_j\varphi_{ij}\Big)\Big]^{1/2}dq_i \tag{1.12}$$

($h, \alpha_2, \ldots, \alpha_n$ are arbitrary constants).

Proof. We will seek a solution of the corresponding equation in the form (1.5). Taking into account (1.10) and (1.11) and using the identity

$$\sum_{i=1}^{n} \varphi_{i1} A_i = 1$$

we obtain the equation

$$\sum_{i=1}^{n} A_i \left[\frac{1}{a_i} \left(\frac{\partial V}{\partial q_i} \right)^2 - b_i - 2U_i + 2 \sum_{j=1}^{k} c_j \Phi_{ij} - 2h\varphi_{i1} \right] = 0$$

whose solution is (1.12). It remains to confirm condition (1.8). Here we will assume that $\alpha_1 = h$. For the solution V obtained we have

$$\text{Det} \left\| \frac{\partial^2 V}{\partial q_i \partial \alpha_j} \right\| = \Delta \cdot 2^{-n+1} \prod_{i=1}^{n} a_i \left(p_i - \sum_{j=1}^{k} \varphi_j \frac{\partial \Phi_j}{\partial q_i} \right)^{-1} \neq 0$$

Corollaries. 1°. If $c_j = 0$ $(j = 1, 2, \ldots, l \leqslant k)$ in (1.11), i.e. $\sigma_j = \varphi_j^{\cdot}$ $(j = 1, 2, \ldots, l)$ and $\Phi_j(q)$ $(j = 1, 2, \ldots, l)$ are arbitrary continuous functions, then the summation over j in the first sum of (1.12) is carried out from $j = l + 1$ to $j = k$.

2°. If all $c_j = 0$, then $\Phi_0 = 0, \Phi_1 = \Phi, \Phi_j = 0$ $(j = 2, 3, \ldots, k)$ leads to the integrable case of /4/.

3°. If $\Phi_j = 0$ $(j = 1, 2, \ldots, k)$ in (1.9), then we have the integrable case of /2/. Putting $\varphi_1 = 1, \varphi_j = 0$ $(j = 2, 3, \ldots, k), c_j = 0$ $(j = 1, 2, \ldots, k)$, $\Phi_0 = 0$, $\gamma = \text{const}$ we arrive at the theorem due to Demin /3/ which generalizes the Stäckel and Moiseyev theorems /5/. The integrable case of Stäckel /1/ corresponds to $\Phi_j \equiv 0$ $(j = 0, 1, \ldots, k)$, $b_i \equiv 0$, $\gamma = \text{const}$, $a_i = \text{const}$ in formula (1.9).

2. Let the motion of a system with n degrees of freedom be described by the equations

$$q_i^{\cdot} = \partial H / \partial p_i, \quad p_i^{\cdot} = -\partial H / \partial q_i + \nu p_i \tag{2.1}$$

$$H = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j + \sum_{i=1}^{n} h^i p_i - U \tag{2.2}$$

where $\nu(t)$ is a given function of time and g^{ij}, h^i, U are functions of the coordinates and time, and $\det \| g^{ij} \| \neq 0$. Here and henceforth the index i will take the values $1, 2, \ldots, n$ unless indicated otherwise.

The differential equations of a number of problems of mechanics can be reduced to the form (2.1). These include, in particular, systems with dissipative forces, some problems of the mechanics of controlled motion, the mechanics of bodies with variable mass and composition in the presence of reactive forces, etc. /6-8/.

Lemma 2.1. Using the substitution

$$p_i = p_i^* \psi(t), \quad \psi(t) = \exp\left(\int \nu dt\right) \tag{2.3}$$

we reduce system (2.1) with Hamiltonian (2.2) to the form

$$q_i^{\cdot} = \partial H^* / \partial p_i^*, \quad p_i^{*\cdot} = -\partial H^* / \partial q_i \tag{2.4}$$

$$H^*(t, q, p^*) = \psi^{-1} H(t, q, p(p^*)) \tag{2.5}$$

Proof. Substituting (2.3) into (2.1) we obtain

$$q_i^{\cdot} = \partial H / \partial p_i, \quad p_i^{*\cdot} = -\psi^{-1} \partial H / \partial q_i \tag{2.6}$$

Using (2.5) we obtain from (2.2)

$$\psi^{-1} \frac{\partial H}{\partial q_i} = \psi \sum_{i,j=1}^{n} A_{ij} p_i^* p_j^* + \sum_{i=1}^{n} B_i p_i^* + \frac{C}{\psi} = \frac{\partial H^*}{\partial q_i} \tag{2.7}$$

where $A_{ij}(t, q)$, $B_i(t, q)$, $C(t, q)$ are the corresponding derivatives of the coefficients of the Hamiltonian (2.2) with respect to the coordinates q_i.

Let us consider the first group of equations in (2.6). Using (2.5) we obtain

$$\frac{\partial H^*}{\partial p_i^*} = \frac{\psi}{2} \sum_{i,j=1}^{n} g^{ij} p_j^* + h^i = \frac{\partial H}{\partial p_i} \tag{2.8}$$

which proves the lemma.

According to Lemma 2.1 a system of the form (2.1) can be reduced to canonical form by substituting the moments (2.3). As a result, the Hamiltonian H^* has the following structure:

$$H^* = \frac{1}{2}\sum_{i,j=1}^{n} g^{*ij} p_i^* p_j^* + \sum_{i=1}^{n} h^{*i} p_i^* - U^* \tag{2.9}$$

$$g^{*ij} = \psi g^{ij}, \quad h^{*i} = h^i, \quad U^* = \psi^{-1} U \tag{2.10}$$

and $\det \| g^{*ij} \| \neq 0$.

Let the canonical system (CS) (2.4) be integrable. Then by virtue of (2.3), (2.10) we can determine in which cases the dynamic systems of the form (2.1) are integrable.

Theorem 2.1. If the CS (2.4) with the Hamiltonian H^* (2.9) is integrable, then so is the system (2.1) with the Hamiltonian H whose coefficients g^{ij}, h^i, U are given by the formulas (2.10), and the general solution of system (2.1) has the form

$$\partial V/\partial \alpha_i = \beta_i, \quad p_i = \psi \partial V/\partial q_i \tag{2.11}$$

where $V(t, q_1, q_2, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$ is the complete integral of the HJ equation of the reduced CS (2.4) and α_i, β_i are arbitrary constants.

Thoerem 2.2. Let the CS (2.4) be autonomous and integrable. Then the non-autonomous system (2.1) will also be integrable, provided that the coefficients g^{ij}, h^i, U and the Hamiltonian H have the form

$$g^{ij} = \psi^{-1}(t) g^{*ij}(q), \quad h^i = h^{*i}(q), \quad U = \psi(t) U^*(q) \tag{2.12}$$

and the general solution of the system (2.1) has the form

$$\partial V/\partial \alpha_i = \beta_i, \quad p_i = \psi \partial V/\partial q_i, \quad V = -\alpha_1 t + W(q, \alpha) \tag{2.13}$$

where $V(t, q, \alpha)$ is the complete integral of the HJ equation of the autonomous CS (2.4).

The proofs of the theorems 2.1 and 2.2 follow from the fact that systems of the form (2.1) can be reduced, in accordance with Lemma 2.1, to canonical form.

Let the structure of the Hamiltonian H be such that $h^i = 0$. Then the following theorem holds for systems (2.1).

Theorem 2.3. Let the CS

$$q_i^{\cdot} = \partial H/\partial p_i, \quad p_i^{\cdot} = -\partial H/\partial q_i \tag{2.14}$$

$$H = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij} p_i p_j - U \tag{2.15}$$

be integrable. Then the system

$$q_i^{\cdot} = \partial H_1/\partial p_i, \quad p_i^{\cdot} = -\partial H_1/\partial q_i + \nu p_i \tag{2.16}$$

$$H_1 = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij} p_i p_j - \gamma(t) U \tag{2.17}$$

will also be integrable provided that the following relation holds:

$$\nu = \gamma^{\cdot}/(2\gamma) \tag{2.18}$$

Proof. Let the HJ equation of the system (2.14)

$$\frac{1}{2}\sum_{i,j=1}^{n} g^{ij} p_i p_j - U + p = 0 \quad \left(p = \frac{\partial S}{\partial t}, \; p_i = \frac{\partial S}{\partial q_i}\right) \tag{2.19}$$

be integrable by the method of separation of variables and have the complete integral $S = S_0(t, \alpha) + W(q, \alpha)$. According to Lemma 2.1 system (2.16) can be reduced to canonical form and the corresponding HJ equation has the form

$$\frac{1}{2}\sum_{i,j=1}^{n} g^{ij} p_i^* p_j^* - \frac{\gamma}{\psi^2} U + \frac{1}{\psi} p^* = 0 \quad \left(p^* = \frac{\partial V}{\partial t}, \; p_i^* = \frac{\partial V}{\partial q_i}\right) \tag{2.20}$$

If condition (2.18) holds then, by virtue of the integrability of Eq.(2.19), Eq.(2.20) whose complete integral is

$$V = \int \sqrt{\gamma}\, \frac{\partial S_0}{\partial t}\, dt + W(q, \alpha) \tag{2.21}$$

is also integrable, and the general solution of system (2.16) has the form

$$\partial V/\partial \alpha_i = \beta_i, \quad p_i = \sqrt{\gamma}\partial W/\partial q_i \tag{2.22}$$

Corollary. If system (2.14) is autonomous: $g^{ij} = g^{ij}(q)$, $U = U(q)$, $H = h = \text{const}$ and integrable, and the complete integral $S = -ht + W(q, h, \alpha)$, then the non-autonomous system (2.16) is also integrable and the complete integral

$$V = -h\int \sqrt{\gamma}\, dt + W(q, h, \alpha)$$

The converse theorem can also be proved.

Theorem 2.4. Let system (2.16) with the Hamiltonian (2.15) be integrable. The the CS (2.14) with the Hamiltonian (2.17) where the quantity γ is replaced by $1/\gamma$ is also integrable, provided that the following relation holds:

$$\gamma = \exp\left(2\int \nu\, dt\right) \tag{2.23}$$

Proof. According to Lemma 2.1 system (2.16) can be reduced to canonical form (2.4). Let the corresponding HJ equation

$$\frac{\psi}{2}\sum_{i,j=1}^{n} g^{ij} p_i^* p_j^* - \psi^{-1} U + p^* = 0 \quad \left(p^* = \frac{\partial V}{\partial t},\ p_i^* = \frac{\partial V}{\partial q_i}\right) \tag{2.24}$$

be integrable using the method of separation of variables, and possess the complete integral $V = V_0(t, \alpha) + W(q, \alpha)$. The HJ equation for CS (2.14) has the form

$$\frac{1}{2}\sum_{i,j=1}^{n} g^{ij} p_i p_j - \frac{1}{\gamma} U + p = 0 \quad \left(p = \frac{\partial S}{\partial t},\ p_i = \frac{\partial S}{\partial q_i}\right) \tag{2.25}$$

Let condition (2.23) hold. Then, since Eq.(2.24) is integrable, so is Eq.(2.25) whose complete integral is

$$S = \int \exp\left(-\int \nu\, dt\right)\frac{\partial V_0}{\partial t}\, dt + W(q, \alpha)$$

The general solution of system (2.16) has the form

$$\partial V/\partial \alpha_i = \beta_i, \quad p_i = \exp\left(\int \nu dt\right)\partial W/\partial q_i$$

and the general solution of CS (2.14) is

$$\partial S/\partial \alpha_i = \beta_i, \quad p_i = \partial W/\partial q_i$$

Corollary. If system (2.16) can be reduced to the autonomous form $g^{*ij} = g^{*ij}(q)$, $U^* = U^*(q)$, $H^* = h = \text{const}$, is integrable and the complete integral $V = -ht + W(q, \alpha)$, then system (2.14) is integrable and the complete integral

$$S = -h\int \exp\left(-\int \nu dt\right) dt + W(q, \alpha)$$

Theorems 2.3 and 2.4 enable us to compare the canonical systems with systems reducible to canonical form and determine their integrability from the form of the Hamiltonian. The theorems on integrability of the systems which can be reduced to canonical form given above, and the comparison theorems, together provide the basis for separating out the classes of autonomous and non-autonomous integrable systems determined by the coefficients g^{ij}, h^i, U of the Hamiltonian functions and the relations (2.10) connecting them.

3. Let us consider non-autonomous dynamic systems of the form (2.1) of the Liouville and Stäckel type. Let the Hamiltonian of system (2.1) have the form

$$H = \frac{1}{2b}\sum_{i=1}^{n}\frac{1}{a_i}\left(p_i - \sum_{j=1}^{k}\varphi_j\frac{\partial \Phi_j}{\partial q_i}\right)^2 - \sum_{j=1}^{k}\sigma_j\Phi_j - \frac{\gamma}{b}\sum_{i=1}^{n} U_i + \Phi_0 \tag{3.1}$$

where the notation of (1.2) is used. Then the Hamiltonian of the reduced CS (2.4) will have the form

$$H^* = \frac{\psi}{2b}\sum_{i=1}^{n}\frac{1}{a_i}\Big(p_i^* - \sum_{j=1}^{k}\frac{\varphi_j}{\psi}\frac{\partial\Phi_j}{\partial q_i}\Big)^2 - \sum_{j=1}^{k}\frac{\sigma_j}{\psi}\Phi_j - \frac{\gamma}{\psi b}\sum_{i=1}^{n}U_i + \frac{\Phi_0}{\psi} \tag{3.2}$$

Returning now to the results of Theorem 1.1 we shall point out cases when systems (2.1) with the Hamiltonian (3.1) are integrable.

Theorem 3.1. If the following conditions hold for the system (2.1) with the Hamiltonian (3.1):

$$b\Phi_j = \sum_{i=1}^{n}\Phi_{ij}(q_i), \quad \nu = \frac{\gamma^{\cdot}}{2\gamma}, \quad \sigma_j = \varphi_j\frac{d}{dt}\Big(\ln\frac{\varphi_j}{\sqrt{\gamma}}\Big) - c_j\gamma \quad (j = 1, 2, \ldots, k) \tag{3.3}$$

then system (2.1) is integrable and its general solution has the form

$$\partial V/\partial\alpha_i' = \beta_i, \quad p_i = \sqrt{\gamma}\,\partial V/\partial q_i \tag{3.4}$$

where α_i', β_i denote $2n$ arbitrary constants and

$$V = -\int\Big(h\sqrt{\gamma} + \frac{\Phi_0}{\sqrt{\gamma}}\Big)dt + \sum_{i=1}^{n}\int\Big[2a_i\Big(U_i + hb_i - \sum_{j=1}^{k}c_j\Phi_{ij} + \alpha_i\Big)\Big]^{1/2}dq_i + \sum_{j=1}^{k}\frac{\varphi_j}{\sqrt{\gamma}}\Phi_j \tag{3.5}$$

is the complete integral of the reduced CS with the Hamiltonian (3.2), and

$$\alpha_1' = h, \quad \alpha_i' = \alpha_i\ (i = 2, 3, \ldots, n), \alpha_1 = -(\alpha_2 + \alpha_3 + \ldots + \alpha_n) \tag{3.6}$$

Theorem 3.2. Let the following conditions hold for system (2.1) and the Hamiltonian (3.1):

$$\nu = \frac{\gamma^{\cdot}}{2\gamma}, \quad \sigma_j = \varphi_j\frac{d}{dt}\Big(\ln\frac{\varphi_j}{\sqrt{\gamma}}\Big) \quad (j = 1, 2, \ldots, k) \tag{3.7}$$

Then system (2.1) will be integrable, its general solution will have the form (3.4) and the complete integral of the reduced CS with the Hamiltonian (3.2) will have the form (3.5) where all $c_j \equiv 0\ (j = 1, 2, \ldots, k)$ and relations (3.6) also hold.

The proof of the theorems follows from Lemma 2.1 and Theorem 1.1.

Let the Hamiltonian of system (2.1) have the form

$$H = \frac{1}{2}\sum_{i=1}^{n}\Big[\frac{A_i}{a_i}\Big(p_i - \sum_{j=1}^{k}\varphi_j\frac{\partial\Phi_j}{\partial q_i}\Big)^2 - \gamma A_i b_i\Big] - \sum_{j=1}^{k}\sigma_j\Phi_j - \gamma\sum_{i=1}^{n}A_iU_i + \Phi_0 \tag{3.8}$$

where the notation of (1.9) is used. Then the Hamiltonian of the reduced CS will be equal to

$$H^* = \frac{\psi}{2}\sum_{i=1}^{n}\Big[\frac{A_i}{a_i}\Big(p_i^* - \sum_{j=1}^{k}\frac{\varphi_j}{\psi}\frac{\partial\Phi_j}{\partial q_i}\Big)^2 - \frac{\gamma}{\psi^2}A_ib_i\Big] - \sum_{j=1}^{k}\frac{\sigma_j}{\psi}\Phi_j - \frac{\gamma}{\psi}\sum_{i=1}^{n}A_iU_i + \frac{\Phi_0}{\psi} \tag{3.9}$$

The following theorems hold for systems (2.1) with the Hamiltonian (3.8).

Theorem 3.3. Let system (2.1) have a Hamiltonian H of the form (3.8). Then, provided that the conditions

$$\Phi_j = \sum_{i=1}^{n}A_i\Phi_{ij}(q_i), \quad \nu = \frac{\gamma^{\cdot}}{2\gamma}, \quad \sigma_j = \varphi_j\frac{d}{dt}\Big(\ln\frac{\varphi_j}{\sqrt{\gamma}}\Big) - c_j\gamma \quad (j = 1, 2, \ldots, k) \tag{3.10}$$

hold, system (2.1) will be integrable and its general solution will have the form

$$\partial V/\partial\alpha_i = \beta_i, \quad p_i = \sqrt{\gamma}\,\partial V/\partial q_i, \quad \alpha_1 = h \tag{3.11}$$

$$V = -\int \left(h\sqrt{\gamma} + \frac{\Phi_0}{\sqrt{\gamma}}\right) dt + \sum_{i=1}^{n} \int \Big[a_i\Big(b_i + 2U_i + 2h\varphi_{i1} - 2\sum_{j=1}^{k} c_j\Phi_{ij} + \sum_{j=2}^{n} \alpha_j\varphi_{ij}\Big)\Big]^{1/2} dq_i + \sum_{j=1}^{k} \frac{\varphi_j}{\sqrt{\gamma}}\Phi_j \tag{3.12}$$

is the complete integral of the reduced CS with the Hamiltonian (3.9).

Theorem 3.4. Let system (2.1) have a Hamiltonian H of the form (3.8). Then, provided that conditions (3.7) hold, system (2.1) will be integrable, its general solution will have the form (3.11) and the complete integral of the reduced CS with the Hamiltonian (3.9) will have the form (3.12) where all $c_j \equiv 0\ (j = 1, 2, \ldots, k)$.

The proofs of the theorems follow from Lemma 2.1 and Theorem 1.2.

4. Examples. 1°. We shall illustrate the general method of reduction to canonical form and integration of dynamic systems by considering the problem of two bodies (material points) of variable mass, which may find application in celestial mechanics /9/. The bodies attract each other in accordance with Newton's Law and are situated within a gaseous or dust cloud exerting additional "frictional" forces and Hooke's elastic force. The equations of motion have the form

$$\mathbf{r}^{\cdot\cdot} = -\mu(t)\, r^{-3}\mathbf{r} + \nu(t)\,\mathbf{r}^{\cdot} + \varkappa(t)\,\mathbf{r}, \quad \mu(t) = GM(t) \tag{4.1}$$

where G is the gravitational constant, $M(t)$ is the mass of the two bodies, $\nu, \varkappa$ are continuous functions of time characterizing the background, and $\mathbf{r}$ is the radius vector of the motion of one material point relative to the other. Using the spherical coordinates r, φ, λ we can write the equations of motion (4.1) in the form (2.1), using the results of Lemma 2.1. The corresponding HJ equation will have the form

$$\frac{\psi}{2}\left[\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \varphi}\right)^2 + \frac{1}{r^2\cos^2\varphi}\left(\frac{\partial V}{\partial \lambda}\right)^2\right] - \frac{\mu}{\psi r} - \frac{\varkappa}{2\psi} r^2 + \frac{\partial V}{\partial t} = 0 \tag{4.2}$$

Let the conditions

$$\mu/\psi^2 = \mu_0, \quad \varkappa/\psi^2 = \varkappa_0, \quad \nu = \mu^{\cdot}/(2\mu) \tag{4.3}$$

hold. Then Eq.(4.2) can be integrated and its complete integral will have the form

$$V = -\alpha_1 \int \left(\frac{\mu}{\mu_0}\right)^{1/2} dt + \int \left[2\left(\frac{\mu_0}{r} + \frac{\varkappa_0}{2} r^2\right) - \frac{\alpha_2^2}{r^2} + 2\alpha_1\right]^{1/2} dr + \int \left[\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\varphi}\right]^{1/2} d\varphi + \alpha_3\lambda \tag{4.4}$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary constants. The general solution of problem (4.1) will be given by formulas of the form (2.11).

We note that problem (4.1) is integrable when the potential is of a more general type

$$U(r, \varphi, \lambda, t) = \mu(t)\left[f(r) + \frac{\Phi(\varphi)}{r^2} + \frac{\Phi(\lambda)}{r^2\cos^2\varphi}\right] \tag{4.5}$$

where $f(r)$, $\Phi(\varphi)$, $\Phi(\lambda)$ are arbitrary differential functions. Carrying out the same arguments for problem (4.1) with the potential (4.5), we obtain the complete integral V of the corresponding HJ equation

$$V = -\alpha_1 \int \left(\frac{\mu}{\mu_0}\right)^{1/2} dt + \int \left[2\mu_0 f(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1\right]^{1/2} dr + \int \left[\alpha_2^2 - 2\mu_0\Phi(\varphi) - \frac{\alpha_3^2}{\cos^2\varphi}\right]^{1/2} d\varphi + \int [\alpha_3^2 - 2\mu_0\Phi(\lambda)]^{1/2} d\lambda \tag{4.6}$$

The general solution of the problem will be given by formulas (2.11) where the complete integral V has the form (4.6). The solution obtained generalizes the results of /9/ to the case of a potential of general type (4.5) for a gravitating and resisting medium.

2°. Let us consider the motion of a passively gravitating point in a gravitational field of two bodies $M_1(t)$ and $M_2(t)$ of variable mass, moving along a straight line passing through the centres of mass. We take into account the influence of the gravitating medium generating additional forces acting on the material point, analogous to "frictional" forces, and Hooke's elastic force. The motion of the attracting bodies themselves is determined by the Gil'den-Meshcherskii problem which has linear solutions /10/. Let the masses M_1 and M_2 vary with time according to the same law.

The equations of motion of a point in a rectangular system of coordinates with origin at the centre of mass of M_1 and M_2 and x axis collinear with the line of motion of bodies with finite masses, will be written in the form

$$\mathbf{r}^{\cdot\cdot} = \operatorname{grad} U + \nu \mathbf{r}^{\cdot} + \beta \mathbf{r} \tag{4.7}$$

where $\mathbf{r}$ is the radius vector of the point, $\nu(t)$, $\beta(t)$ are continuous functions determining the parameters of the gravitating medium, the potential has the form

$$U = \mu_1/r_1 + \mu_2/r_2, \quad \mu_i = GM_i(t) \quad (i = 1, 2)$$

and r_1, r_2 are the distances between the material point and the attracting bodies.

Using the notation of /4/, we shall write Eqs.(4.7) in the form (2.16) where $i = 1, 2, 3$, and the Hamiltonian has the form

$$H = \frac{2}{r_{12}^2(\lambda^2 - \eta^2)} \Big\{ (\lambda^2 - 1)\Big[p_\lambda - \frac{r_{12}^{\cdot} r_{12}}{4}\frac{\partial v}{\partial \lambda}\Big]^2 +$$
$$(1 - \eta^2)\Big[p_\eta - \frac{r_{12}^{\cdot} r_{12}}{4}\frac{\partial v}{\partial \eta}\Big]^2 + \Big(\frac{1}{\lambda^2 - 1} + \frac{1}{1 - \eta^2}\Big) p_w^2 \Big\} -$$
$$\frac{r_{12}^{\cdot 2}}{4} v - 2\frac{(\mu_1 + \mu_2)\lambda - (\mu_1 - \mu_2)\eta}{r_{12}(\lambda^2 - \eta^2)} - \frac{\beta r_{12}^2}{4} v$$
$$v = \frac{\lambda^2 + \eta^2}{2} + \lambda\eta k + \frac{k^2 - 1}{2}$$

and the coordinates λ, η, w are regarded as generalized coordinates $q_i\ (i = 1, 2, 3)$.

According to Lemma 2.1 system (2.16) can be reduced to the canonical form

$$q_i^{\cdot} = \partial H^*/\partial p_i^*, \quad p_i^{*\cdot} = -\partial H^*/\partial q_i \quad (i = 1, 2, 3) \tag{4.8}$$
$$H^*(t, q, p^*) = \psi^{-1} H(t, q, p(p^*))$$

Let the following conditions hold:

$$\varphi = \frac{r_{12}^{\cdot} r_{12}}{4\psi}, \quad \frac{r_{12}\mu}{\psi^2} = C_0, \quad \frac{r_{12}\mu_i}{\psi^2} = C_i \quad (i = 1, 2) \tag{4.9}$$
$$r_{12}^3 (r_{12}^{\cdot\cdot} - \nu r_{12}^{\cdot} - \beta r_{12})/(8\psi^2) = C; \quad C = 0, \quad k \neq 0; \quad C \neq 0, \quad k = 0$$

The Gil'den-Meshcherskii problem determining the motion of the bodies M_1 and M_2 admits of the following linear solutions /10/:

$$\mathbf{r}_{12} = \mu^{-1}\lambda_0, \quad \lambda_0^3 = -\mu_0/b_0, \quad b_0 \neq 0, \quad b_1 = 0 \tag{4.10}$$
$$\mathbf{r}_{12} = (3b_1\zeta)^{2/3}\mu^{-1}\lambda_0, \quad b_1 \neq 0; \quad \zeta = \int \mu^2\, dt$$

Assuming $\mu(t)$ to be given, we can determine the function $\nu(t), \beta(t), \varphi(t), \psi(t)$, according to /10/, from the conditions of integrability of (4.9), taking (4.10) into account. As a result we obtain for $b_1 = 0$

$$r_{12} = \mu^{-1}\lambda_0, \quad \nu = 0, \quad \psi = \sqrt{\lambda_0/C_0} \tag{4.11}$$
$$\beta = (b_0 - 8CC_0^{-1}\lambda_0^{-3})\mu^4, \quad \varphi = -{}^1/_4 \sqrt{C_0}\, \lambda_0^{3/2} \mu^{\cdot} \mu^{-3}$$

and for $b_1 \neq 0$

$$r_{12} = (3b_1\zeta)^{2/3}\mu^{-1}\lambda_0, \quad \nu = {}^1/_3 \mu^2 \zeta^{-1}, \quad \psi = \sqrt{\lambda_0/C_0}\,(3b_1\zeta)^{1/3} \tag{4.12}$$
$$\beta = (-2b_1^2 + b_0 - 8CC_0^{-1}\lambda_0^{-3})(9b_1^2)^{-1}\mu^4\zeta^{-2} + \nu\mu^{\cdot}\mu^{-1}$$
$$\varphi = {}^1/_4 \sqrt{C_0}\,[2b_1 - \mu^{\cdot}\mu^{-3} 3b_1\zeta]\,\lambda_0^{3/2}$$

Formulas (4.11) and (4.12) determine two classes of solutions of the bounded linear problem of three bodies of variable mass, taking into account the gravitating and resistive medium.

The complete integral of the HJ equation of the CS (4.8) has the form

$$V = \varphi v - h\int \frac{\psi}{r_{12}^2}\, dt + \int \frac{\sqrt{Q_+(\lambda)}}{\lambda^2 - 1}\, d\lambda + \int \frac{\sqrt{Q_-(\eta)}}{1 - \eta^2}\, d\eta + \alpha_3 w$$
$$Q_\pm(\lambda) = (\lambda^2 - 1)\Big[-\frac{C}{2}\lambda^4 + \frac{h + C}{2}\lambda^2 + (C_1 \pm C_2)\lambda + \alpha_2\Big] - \alpha_3^2$$

and $C = 0$ when $k \neq 0$; $C \neq 0, C_1 = C_2$ when $k = 0$; h, α_2, α_3 are arbitrary constants. The functions φ, r_{12}, ψ are given by the formulas (4.11) or (4.12).

The general solution of the problem has the form (β_i are arbitrary constants)

$$\partial V/\partial \alpha_i = \beta_i, \quad p_i = \psi(t)\, \partial V/\partial q_i \quad (i = 1, 2, 3), \quad \alpha_1 = h$$

Problem (4.7) was studied for the case when $\nu = 0, \beta = 0$ in /4, 11/. A spatial-temporal transformation method was used in /11/ to show that the problem can be solved when the mass $\mu(t)$ varies according to the first Meshcherskii law, and in the case of the generalized Meshcherskii law it has a solution when the masses are equal to each other $\mu_1 = \mu_2$. A solution of the problem was obtained in /4/ using the Jacobi method for the case when the mass change obeys the first Meshcherskii law.

We shall now state the following, more general result. Assuming that the functions $\mu(t), r_{12}(t)$ are given, we obtain from the conditions of integrability (4.9) the expressions

$$\nu(t) = {}^1/_2 (r_{12}^{\cdot}/r_{12} + \mu^{\cdot}/\mu) \tag{4.13}$$
$$\beta(t) = r_{12}^{\cdot\cdot}/r_{12} - {}^1/_2 (r_{12}^{\cdot}/r_{12} + \mu^{\cdot}/\mu)\, r_{12}^{\cdot}/r_{12} - 8CC_0^{-1}\mu/r_{12}^3$$

and in (4.13) $C = 0$ when $k \neq 0$. This implies that the equations of motion (4.7) can be integrated for any specified, continuously differentiable functions $\mu(t), r_{12}(t)$ which do not vanish within the time interval in question, provided that $\nu(t), \beta(t)$ can be found from formulas (4.13).

REFERENCES

1. DUBOSHIN G.N., Celestial Mechanics: Basic Problems and Methods. Moscow, Nauka, 1975.
2. YAROV-YAROVOI M.S., On integrating the Hamilton-Jacobi equation by the method of separation of variables. PMM, 27, 6, 1963.
3. DEMIN V.G., Motion of an Artificial Satellite in a Non-central Gravitational Field. Moscow, Nauka, 1968.
4. BEKOV A.A. and OMAROV T.B., Integrable cases of the Hamilton-Jacobi equation and certain non-steady problems of celestial mechanics. Astron. zhurn, 55, 3, 1978.
5. MUL'TON F., Introduction to Celestial Mechanics. Moscow, Glav. red. obshchetekhn. lit. i nomogr. ONTI, 1935.
6. MESHCHERSKII I.V., Papers on the Mechanics of Bodies of Variable Mass. Moscow-Leningrad, Gostekhizdat, 1949.
7. NOVOSELOV V.A., Analytical Mechanics of Systems with Variable Mass. Leningrad, Izd-vo LGU, 1969.
8. BERKOVICH L.M. and GEL'FGAT B.E., Investigation of certain non-steady problems of celestial mechanics using transformation methods. In: Problems of Analytical Mechanics, Theory of Stability and Control. Moscow, Nauka, 1975.
9. OMAROV T.B., Dynamics of Gravitating Systems of the Metagalaxy. Alma-Ata, Nauka, 1975.
10. BERKOVICH L.M., Transformation of the Gil'den-Meshcherskii problem to its stationary form and the laws of mass change. PMM 44, 2, 1980.
11. GEL'FGAT B.E., Integrable cases of the problem of three bodies of variable mass. In: Problems of the Mechanics of Controlled Motion. Perm', Izd-ye Permsk. un-ta, 4,

Translated by L.K.